Heliodyne Corporation

Research Report 8 December 1962

Mixing and Reactions in a Turbulent Wake

Andrew F. Probstein

RESEARCH REPORT 8

MIXING AND FLUCTUATIONS IN A TURBULENT WAKE

by Andrew P. Proudian

HELIODYNE CORPORATION
Los Angeles, California 90064

December, 1964

supported by

ADVANCED RESEARCH PROJECTS AGENCY

under ARPA Order 360
monitored by the
U.S. ARMY ORDNANCE MISSILE COMMAND
Redstone Arsenal, Alabama

under Contract No. DA 04-495-AMC-458(Z)

This research is a part of Project DEFENDER, sponsored by the Advanced Research Projects Agency, Department of Defense.

RR8, 12-64

ABSTRACT

The structure of turbulent wakes of hypersonic bodies is argued to be largely controlled by the combined action of relatively large scale random convective motions, which introduce inhomogeneities in the wake core, and of molecular diffusion aided by turbulent distortions of fluid elements. A previously developed mathematical model representing turbulent wake mixing in terms of a mixing lag is discussed and qualitatively justified in terms of the above described mixing mechanism, and somewhat generalized. Numerical predictions for the magnitude and variation of mass density fluctuations in a relatively low speed wake are obtained by solving a simplified set of equations, in which the lag is represented by an effective mixing boundary, and are in encouraging agreement with the general magnitude and trend observed experimentally, although firm experimental measurements are lacking to-date.

TABLE OF CONTENTS

1. INTRODUCTION

The study of the structure of turbulent wakes of hypersonic bodies has been pursued vigorously in recent years as part of the continuing research in re-entry phenomenology. Many theoretical analyses of the turbulent wake[1, 2, 3] have been based on an extension of laminar wake analyses, through the use of turbulent transport coefficients to represent the effects of turbulence. The proper characterization of the variation of fluid properties and of chemical reactions within the turbulent wake requires, however, a more detailed description of turbulent mixing than is provided by analyses based on eddy transport coefficients, which are particularly deficient in representing the effects of large local inhomogeneities on chemical reactions in the turbulent core.

In an attempt to provide a more physical description of the important features of mixing in the turbulent wake, a simple wake mixing model, based on the known properties of turbulent convection and mixing in low speed wakes has been previously formulated[4, 5]. In that model, a distinction was made between a fluid component consisting of fluid ingested into the turbulent core and randomly convected within it, but which had not undergone appreciable mixing or diffusion on a molecular scale and a more homogeneous well mixed component. A mixing lag was used to represent the delay in molecular mixing of fluid newly ingested into the turbulent core. The present paper reviews and discusses the physical basis and validity of the model in somewhat greater detail, proposes a somewhat more general formulation of wake mixing and presents a numerical solution to a simplified set of equations representing the

mixing model, in which the mixing lag previously formulated is represented by a mixing boundary which "lags" behind the turbulent boundary.

The numerical solution was obtained for the wake of a relatively low speed blunt body, representative of the speeds of pellets fired in ballistic ranges. The results are in at least qualitative agreement with experiment[6,7], insofar as predictions of root mean square gas density fluctuations in the wake are concerned. Unfortunately, the absolute magnitude of the fluctuations observed experimentally is still somewhat in doubt at this time. Additionally, it must be noted that while the interest in hypersonic turbulent wakes is centered on the case of high speed wakes, mass density fluctuation measurements have so far been confined to relatively low velocities. Thus, the most striking effect of the present model, which is its influence on the wake chemistry, and particularly the ionization chemistry, cannot be directly tested at this time. However, the ability of the model to reproduce the gross features of the mass density fluctuations at the lower speeds is an encouraging indication that it may be applied with some confidence to describe the properties of hypersonic re-entry wakes.

2. PHENOMENOLOGY OF TURBULENT WAKE MIXING AND WAKE STRUCTURE

The growth of the turbulent wake behind bodies in high Reynolds number flows, and the processes of engulfment of the outer flow in the wake growth, and of mixing and convection of the core gases are relatively well understood for subsonic flows[8, 9]. While there undoubtedly will exist differences between such wakes and the wakes of hypersonic bodies, it can be reasonably assumed that the same qualitative features will persist in the hypersonic case, particularly when it is realized that the velocity fluctuations in such wakes are subsonic beyond the first few body diameters, due to the very rapid initial decay of the mean wake velocity. On the assumption that low speed turbulence results are applicable to hypersonic wakes, the growth of the turbulent wake and the manner in which it engulfs outer fluid may be broadly characterized as follows: The local propagation of the turbulent front bounding the wake core occurs by diffusion of vorticity fluctuations into the outer fluid[9, 10]. The diffusion is initiated by the smallest eddies, since they represent the greatest velocity gradients. The transfer of turbulence energy to the previously non-turbulent fluid is accomplished by the energy containing eddies, which transfer energy to the vorticity diffused into the outer fluid by the small eddies. In the above manner, turbulence is generated into previously quiescent fluid. The turbulent wake "front" represents a boundary between fluid which possesses vorticity and vorticity fluctuations, and fluid which does not. Corrsin and Kistler[10] describe the front as a "laminar superlayer", of very small thickness, across which vorticity decreases from values

characteristic of the turbulent flow, to very low values. Vorticity is communicated locally to new fluid across this thin layer, principally by the smallest eddies of the turbulent flow, resulting in a "propagation" of the front into new fluid. The new fluid thus acquires vorticity, and vorticity fluctuations, but it acquires no large scale random motions by the above mechanism. Consequently, the newly engulfed fluid would tend to remain close to its non-turbulent "trajectory", under the effect of the primary turbulence propagation mechanism. It would therefore not be really mixed as yet, and the small scale motions induced in it would not appreciably aid the molecular mixing process.

If there were no other mixing mechanism, each additional "layer" of engulfed fluid would come in contact with only the preceding layer of engulfed fluid, and effective mixing would not occur. Effective mixing would require a mechanism which swept away newly engulfed fluid and mixed into the "deeper" parts of the flow, while exposing new layers to fluid brought up from those deeper parts. Such a mechanism does indeed exist, but it is quite distinct from the primary entrainment mechanism, and is due to the turbulent motions on another scale. Indeed, in addition to the small scale eddies, the turbulent flow exhibits larger scale eddy motions, which convect turbulent fluid from one part of the wake to another. (The large eddies also serve to contort the turbulent front and thereby greatly increase the rate of entrainment of new fluid, since the latter depends on the total area of the "laminar superlayer".) The newly engulfed fluid will not become mixed into the core until it is convected by one of those larger eddy motions. The scale of the inhomogeneities of density, temperature, etc. introduced into the

turbulent core in this manner then depends on how much new fluid is engulfed by the propagation of the turbulent front. It may be roughly estimated as follows: The main convective action of the turbulence is carried on by the larger eddies of the turbulent flow, of scale L, say. The time required by those eddies to sweep away newly engulfed fluid is of the order $\tau \sim (L/u')$, where u' is the mean velocity of the larger eddies. Over a period of time τ the turbulent front will propagate a distance $D \sim V^* \tau$, where V^* is the mean propagation velocity of the front. The "transverse" scale of the engulfed fluid will be of the order of D, while its "longitudinal" scale (along the boundary) will be some fraction of L. These scales may now be compared with the scales of inhomogeneities at which molecular diffusion becomes dominant. The latter scale is comparable to the dissipation or Kolmogorov scale when the kinematic viscosity and diffusivity are comparable. In a well developed turbulent flow, it can be expected to be two or more orders of magnitude below the macroscale of turbulence, namely L. The question remains whether D is also larger than the Kolmogorov scale. From the above discussions, we may write

$$D \sim V^* \tau \sim L \frac{V^*}{u'} = L\left(\frac{V^*}{V_\infty}\right)\left(\frac{V_\infty}{\Delta V}\right)\left(\frac{\Delta V}{u'}\right) \qquad (1)$$

where V_∞ is the free stream velocity, and ΔV is the velocity difference across the wake. The ratio $(V_\infty/\Delta V)$ of the free stream velocity to the wake velocity defect can be roughly taken as $(V_\infty/\Delta V) \sim 8y_f^2 (\rho/\rho_\infty)$ where y_f is the non-dimensional wake radius, normalized with respect to the body diameter and (ρ/ρ_∞) is the ratio of mean wake to

free stream density. The ratio $(\Delta V/u')$ is approximately 2, according to subsonic turbulent wake results. The ratio (V^*/V_∞) is approximately the same as the wake "slope" (dy_f/dz). Assuming an approximate (non-dimensional) wake growth law of the form $y_f \sim 0.5z^{1/3}$, where z is downstream distance expressed in body diameters, the above estimates can be substituted in Eq. (1), which yields:

$$D = \frac{2}{3} L\left(\frac{\rho}{\rho_\infty}\right) . \qquad (2)$$

In the region beyond the first ten or twenty body diameters where the present model can be expected to apply, the mean density ρ in the wake will not be less than a few times below the free stream density, so that D will be of the order of one-tenth of L, which may reasonably be assumed to be significantly greater than the Kolmogorov scale. Thus it appears reasonable to assume that the combined effects of the propagation of the turbulent front and the convective action of the large eddies of the turbulence tend to introduce into the wake fluid elements of a scale large compared to the Kolmogorov or dissipative scale of the turbulence, so that these must be reduced in scale by the cascading process of turbulence before effective molecular mixing can occur. The relative time scales for the turbulent cascade process and molecular diffusion or conduction are discussed in Appendix B. It is shown in that Appendix that for large Reynolds numbers, the critical assumption that the turbulent cascade time is short compared to the diffusion time scale is valid.

In summary, the engulfment of the outer wake by the turbulent core may be pictured as a process whereby relatively large fluid

elements (i.e., large compared to the Kolmogorov scale) are convected into the turbulent core by the action of large scale eddies, while at the same time they are distorted by the shearing effects of small scale eddy motions[11] until the scale of the inhomogeneities has been reduced (and the gradients steepened) to the point where molecular diffusion erases those inhomogeneities and mixes the engulfed fluid with the core fluid on a molecular scale.*

As a result of the mixing process described above, the wake can be expected to consist essentially of two distinguishable components, namely a component which has not been mixed on a molecular scale with the "main" flow, and whose properties are close to the properties it would have if it were not engulfed in the core, and a more or less homogeneous component which consists of the initial core fluid augmented by that portion of the engulfed fluid which has been molecularly mixed. Since both the entrainment process and the mixing process are random in nature, the above description of the wake must be assumed to apply 'in the mean,' so to speak.

The model for the wake structure formulated below assumes that there is no significant fluid component which is neither unmixed nor completely (molecularly) mixed. This assumption is more important when chemical reactions are concerned, in which case the times characteristic of chemical reactions must be long compared

---

*In the absence of molecular diffusion and conduction, the effect of eddy diffusivity (or random convection) is to break up large volume elements into smaller ones (down to the cut-off scale) and, therefore, to increase mean gradients rather than decrease them. The increase in mean gradients leads, however, to increased molecular dissipation, so that eddy diffusion indirectly leads to dissipation of inhomogeneities.

to the time required for fluid elements to 'jump' across the Kolmogorov scale. However, the basic concept used in the model, namely that a macroscopic engulfment and mixing precedes molecular mixing, is not dependent upon that assumption.

In the model of wake structure described above, the spatial variations in wake properties, such as mass density, electron density, and temperature, are regarded as arising from the existence of 'unmixed' fluid elements intermingled with 'mixed' fluid, as well as from inhomogeneities which may exist within or between unmixed fluid elements. Inhomogeneities in the mixed component are assumed small, and will in fact be neglected in the following.

The mean density fluctuations which can be predicted on the basis of a two-component wake model such as the one described above are in semi-quantitative agreement as will be shown below, with the large mass density fluctuations apparently found[6, 7, 12] in schlieren measurements of density fluctuations in the wakes of hypervelocity pellets at large downstream distances from the pellets. The apparent magnitude of the measured fluctuations at large downstream distances, of the order of at least one tenth at an axial distance of several hundred body diameters, would seem to rule out the possibility that the fluctuations are created within the turbulent flow by compressibility and dissipative effects. Indeed it is shown in Appendix A that such effects would lead to considerably smaller fluctuations. Similarly, interaction between the turbulence and the wake chemistry must be ruled out for the relatively low temperatures in the observed ballistic range wakes.

In the far wake, where the turbulent velocities have presumably died down, the above model is not expected to hold. The eddy diffusivity is probably no longer very important, and the break up of the lumps no longer occurs very efficiently. The temperature inhomogeneities then decay essentially by conduction. In fact, sufficiently far downstream all velocities will have essentially died down. Thus, the final steps of thermal equalization proceed purely by molecular diffusion. The limit of application of the 'two-fluid' model is undoubtedly dependent on Reynolds number and body geometry, and it may not be applicable at all when the turbulent intensity is very low, corresponding to small Reynolds numbers.

The model may also require modification for application to the wakes of re-entry bodies, particularly close to the body, as the character of the turbulence in the near wake may be quite different, and compressibility effects in particular may be important.

3. WAKE MIXING EQUATIONS

A simple set of equations describing a quasi one-dimensional wake with a mean turbulent boundary, and incorporating the physical features of wake mixing discussed in the previous section, is presented below. The equations are similar to those presented in References 4 and 5. As in those references, the growth of the (mean) turbulent wake is represented by the flux of inviscid fluid into the turbulent core through the wake boundary. The entering fluid then forms part of the unmixed portion of the core for a mean distance ξ (measured in body diameters), after which it is homogeneously mixed with the remainder of the wake by molecular effects. The actual molecular mixing of fluid entering at a particular station in the wake need not occur suddenly after a distance ξ, but may be spread over a (small) range of lags centered about ξ. This is discussed further in the following section.

The turbulent wake is then characterized by the pressure (assumed constant across the wake), and by the velocity, density, enthalpy and chemical composition (if chemistry is included) of the homogeneously mixed and unmixed portions of the wake, together with the relative fractions of each, as a function of downstream distance. The values of the velocity, density, enthalpy, and chemical composition of the inviscid gas at the edge of the turbulent core must be assumed to be specified. The wake evolution can then be described by the one-dimensional equations of conservation of mass, momentum and energy, by an equation describing the mixing lag, and by the equations expressing the chemical reactions or the thermodynamic characteristics of the gas (equations of state) if chemical

reactions can be ignored. In the latter case, which is applicable to wakes of relatively low-speed projectiles in ballistic ranges for instance, the conservation equations specifying the quasi one-dimensional model can be expressed as follows (z denotes downstream distance):

Conservation of Mass:

$$\frac{d}{dz}\int_{A_u}(\rho_u V_u\, dA) + \frac{d}{dz}(\rho_m V_m A_m) = \rho_i V_i \frac{dA}{dz} \tag{3}$$

Conservation of Momentum:

$$\frac{d}{dz}\int_{A_u}(\rho_u V_u^2\, dA) + \frac{d}{dz}(\rho_m V_m^2 A_m) - A\frac{dp}{dz} = \rho_i V_i^2 \frac{dA}{dz} \tag{4}$$

Conservation of Energy:

$$\frac{d}{dz}\int_{A_u}\left\{\rho_u V_u\left(h_u + \frac{V_u^2}{2}\right)dA\right\} + \frac{d}{dz}\left\{\rho_m V_m\left(h_m + \frac{V_m^2}{2}\right)A_m\right\}$$

$$= \rho_i V_i H \frac{dA}{dz} \tag{5}$$

The foregoing equations are the usual conservation equations for one-dimensional flow in a channel with mass addition at the boundary (at the same pressure as in the channel), for a flow consisting of two components. In the foregoing equations, ρ_i is the density of the

inviscid gas at the edge of the turbulent core, V_i is its velocity, H its total enthalpy, $H = [h_i + (V_i^2/2)]$, and p is the pressure. The density ρ, velocity V, and static enthalpy h, for the mixed component and for the unmixed portions of the turbulent wake are denoted by the subscripts m and u respectively. The areas A_m and A_u are the 'partial areas' of hot and cold fluid, and their sum is equal to the wake area A:

$$A_m + A_u = A \tag{6}$$

In the above cited references, it had been assumed that the unmixed fluid could be characterized by a single average value of ρ and V. This assumption is not necessary in the calculations presented below, in which the properties of the unmixed fluid component are assumed known.

The mixing lag can be introduced in the mixing model by an equation of the form

$$\frac{d}{dz}(\rho_m V_m A_m) = \rho_i(z - \xi)V_i(z - \xi)\frac{dA(z - \xi)}{dz} \tag{7}$$

where ξ is the lag distance defined above, which expresses the assumption that fluid ingested at station $(z - \xi)$ becomes part of the mixed component at station z, that is, with a lag ξ.

The solution to the wake mixing equations, Eqs. (3) to (7) becomes particularly simple if it is assumed that the unmixed fluid component is entirely unaffected in its thermodynamic and chemical properties by the random convection to which it is subjected within

the turbulent core. Only the mixed fluid properties then remain to be specified. Equation (7) then shows that in computing the properties of that component, the wake mixing model is equivalent to an instantaneous mixing model[13], but with an effective mixing boundary. The relation of the effective mixing boundary to the mean turbulent boundary is depicted in Fig. (1). Inviscid fluid entering the turbulent boundary at (z_f', y_f') survives molecular mixing for a distance ξ, so that it would have reached the point (z_m, y_m) if it had continued along its laminar trajectory. Since the random convection by the turbulent velocity field in the wake is assumed not to affect the fluid, it is effectively mixed at (z_m, y_m). Thus, the mixing boundary can be constructed from the mean wake boundary by simply following fluid elements on an extension of the laminar streamlines from their point of crossing of the turbulent boundary to a point ξ downstream. The amount of unmixed fluid is then that contained between y_f and y_m.

The wake mixing equations describing the mixed fluid component are then simply the conservation equations for one-dimensional channel flow, with or without chemical reactions, and with mass injection at the effective mixing boundary. In order to solve those equations, the lag must be specified, and it is discussed in the following section.

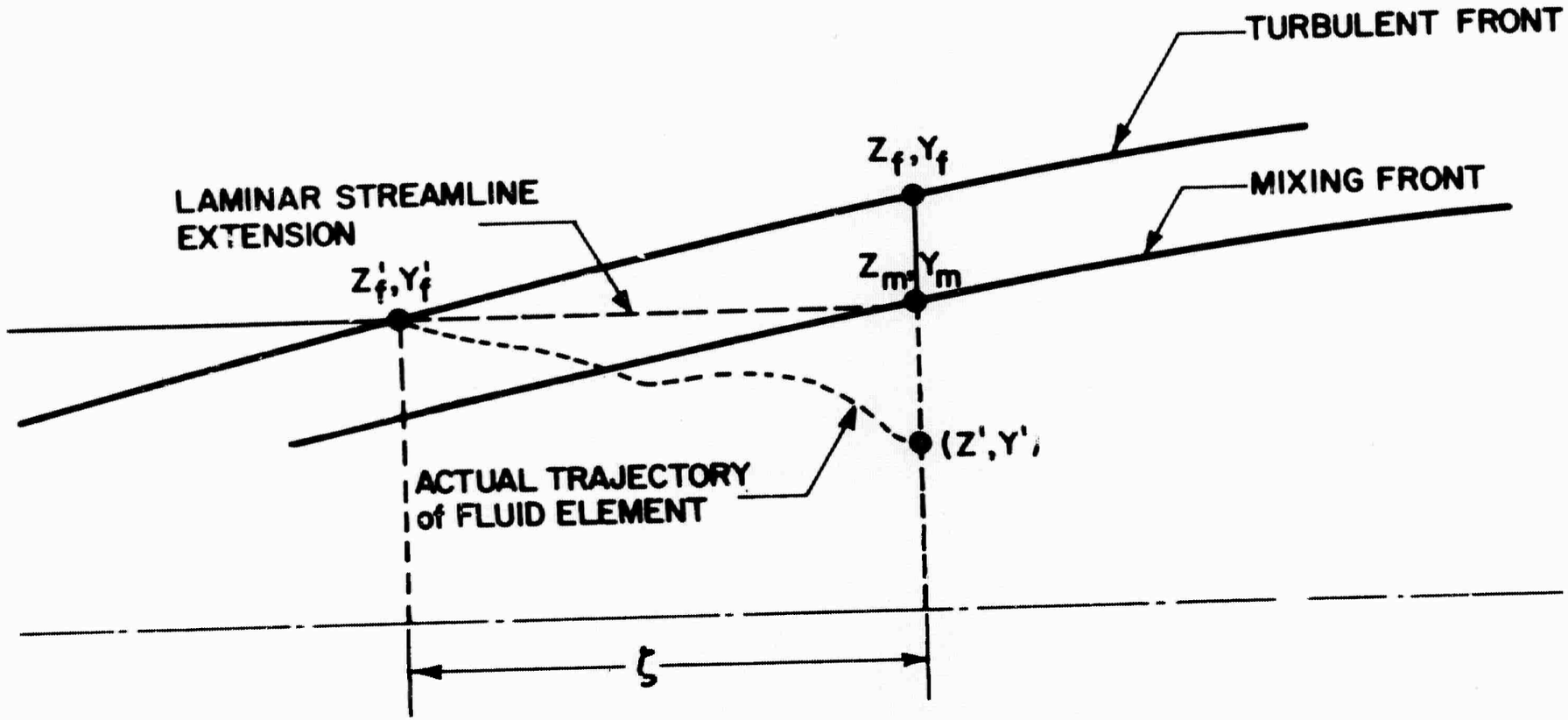

Fig. 1 Effective Mixing Boundary. A fluid element entering the turbulent core at (z'_f, y'_f) is assumed to survive a distance ξ and thus effectively mix at (z_m, y_m), although its actual location may be (z', y').

4. MIXING LAG DISTANCE

In Eq. (7), the assumed lag in the molecular mixing of ingested fluid into the core is represented by a lag distance ξ. This distance is approximately equal to some lag time τ multiplied by the mean wake velocity. The lag time τ is clearly related to the time scale for dissipation of inhomogeneities in a turbulent flow. This lag is determined by the rate of molecular mixing in a turbulent flow field, and is therefore related to the rate of destruction of scalar inhomogeneities in such a flow field. The only such rate presently available from the theory of turbulence is that for the mixing of a passive scalar in a homogeneous steady state field of turbulence[9, 14] and even then the rate is only known at large values of the time, when the effect of the initial distribution of scalar inhomogeneities has disappeared. In that case, the (logarithmic) rate for the dissipation of scalar inhomogeneities is the same as the dissipation rate of kinetic energy. There is, of course, no good a priori reason to believe that the distribution of inhomogeneities in the wake mixing problem would be the same as that in such a homogeneous steady-state field of turbulence. However, it is not unreasonable to hope that the mean rate of dissipation of inhomogeneities predicted for steady-state homogeneous turbulence will not be too far off in the present problem.

The time scale for dissipation of inhomogeneities in homogeneous turbulence can be estimated by using a relation due to Onsager[12] describing the process of turbulent cascade of energy from large scale to small scale eddies:

$$\frac{dk}{k} = \frac{dt}{t_k} , \tag{8}$$

where $t_k \simeq \epsilon^{-1/3} k^{-2/3}$ is the "lifetime" of an eddy of wave number k. Eq. (8) describes an accelerated transfer of energy from large to small scale eddies. In that equation, ϵ is the rate of dissipation of kinetic energy of turbulence by viscous effects.

Equation (8) can be integrated to obtain the characteristic dissipation time scale τ required for the breakup of an inhomogeneity of characteristic scale $D = 1/k_D$ down to the dissipation or Kolmogorov scale $\eta = 1/k_d$, on the assumption that the dissipation of scalar inhomogeneities can be described in terms of the turbulent cascade process. Thus, Eq. (8) yields

$$\frac{3}{2}\int_{T-\tau}^{T} \epsilon^{1/3}\, dt = \int_{k_D}^{k_d} k^{-2/3} \frac{dk}{k} = k_D^{-2/3} - k_d^{-2/3} . \tag{9}$$

Since the Kolmogorov scale k_d is much larger than k_D, the term $k_d^{-2/3}$ can be neglected in the above equation. Furthermore, the dissipation rate ϵ can be written approximately[9] as:

$$\epsilon = \frac{3}{2}\frac{du'^2}{dt} \simeq \frac{3}{2}\frac{u'^3}{L} \tag{10}$$

where $3/2\, u'^2$ is the mean turbulent kinetic energy, and L is the scale of the energy containing eddies of turbulence. Using the above

equation, and assuming that the initial scale of the inhomogeneities is given by Eq. (2), Eq. (9) can be rewritten, approximately:

$$\int_{T-\tau}^{T} -\frac{1}{u'^2}\frac{du'^2}{dt}\,dt = \left(\frac{\bar{\rho}}{\rho_\infty}\right)^{2/3} \tag{11}$$

where $\bar{\rho}$ is a mean wake velocity at the initial time.

Substituting downstream distance for time, Eq. (11) yields:

$$\int_{z-\xi}^{z} -\frac{1}{u'^2}\frac{du'^2}{dz}\,dz = \left(\frac{\bar{\rho}}{\rho_\infty}\right)^{2/3} \tag{12}$$

where $\bar{\rho}$ is to be evaluated at $(z - \xi)$.

Finally, the turbulent velocity can be expressed in terms of the mean wake and full stream velocities, by writing

$$\left(\frac{u'^2}{V_\infty^2}\right) = \left(\frac{u'^2}{\Delta V^2}\right)\left(\frac{\Delta V^2}{V_\infty^2}\right) \quad . \tag{13}$$

Let

$$\frac{u'^2}{\Delta V^2} = \Omega(z)$$

and

$$\left(\frac{\Delta V}{V_\infty}\right)^2 = G(z) .$$

Then Eq. (12) can be immediately integrated to give:

$$G(z) = G(z - \xi)\,\frac{\Omega(z - \xi)}{\Omega(z)}\,\exp\left\{-\left(\frac{\bar{\rho}}{\rho_\infty}\right)^{2/3}\right\} . \qquad (14)$$

The function $\Omega(z)$, which is the ratio of turbulent to mean flow kinetic energy in the wake, tends in subsonic wakes to a constant value at large downstream distances, of the order of a few hundred body diameters for two-dimensional wakes and probably much sooner for axi-symmetric wakes. On the other hand, at small downstream distances, at which $\Omega(z)$ decays fairly rapidly, ξ is a small quantity, compared to the decay scale of $\Omega(z)$. Thus, the ratio $[\Omega(z - \xi)]/[\Omega(z)]$ will be approximately constant and equal to unity everywhere in the wake. With this assumption, Eq. (14) becomes a simple implicit equation for ξ, which depends essentially only on the form of the function $G(z)$ representing the decay of mean wake velocity with downstream distance. It is not sensitive to the exact value of either the macroscale or the dissipation scale of the turbulence, nor to the exact form of the function $\Omega(z)$.

If a power law decay is assumed for the wake velocity, of the form $G(z) = \text{const.} \times z^{-n}$, Eq. (14) yields

$$\xi(z) = z\left[1 - \exp\left\{-\frac{1}{n}\left(\frac{\rho}{\rho_\infty}\right)^{2/3}\right\}\right] \qquad (15)$$

This form has been fitted in the numerical calculations presented below.

Clearly one cannot expect Eq. (14) or (15) to provide an exact estimate of the lag. In fact, such an exact estimate is not very meaningful, since ξ is intended to represent some mean mixing lag, with the actual mixing of any given fluid element occurring over some range of downstream distances. Such a range in mixing distance could be expected to arise from the fact that the survival time τ of a fluid element will exhibit a range of values, from the fact that a given element will not mix entirely at one time, and finally from the fact that there is a velocity profile in the wake, leading to a range in ξ even for a fixed survival time. However, if that range is fairly small compared to the length scale over which the outer (inviscid) field properties vary appreciably, a "sharp" lag ξ can be used to represent mixing of a fluid element over a range of distance after it is ingested. Indeed, a more general form of Eq. (7) representing mixing over a range of downstream distance might be given as*

*The use of a lag equation summing the contributions to the mixed flow at a given station from the unmixed flux at all preceding stations was suggested by Dr. A. Nayfeh of Heliodyne, though in a somewhat different form.

$$\frac{d}{dz}\left(\rho_m V_m A_m\right) = \int_{z_0}^{z} \rho_i(z')V_i(z') \frac{d\,A(z')}{dz'} \frac{\partial \phi(z, z')}{\partial z}\, dz' , \qquad (16)$$

where $\phi(z, z')$ is the portion of the inviscid flux per unit area entering the core at z' which has survived to station z, and therefore $\partial\phi/\partial z$ is a rate of mixing at station z of inviscid fluid which entered the core at z'. (Obviously $\phi \equiv 0$ when $z < z'$.)

The discussion of mixing in a turbulent flow presented in the preceding sections suggests that $\partial\phi/\partial z$ is a function which peaks at some particular value of $z - z'$, namely when the shredding effect of turbulence has reduced the scale of the inhomogeneities to the Kolmogorov scale, and is small everywhere else.

Thus, particularly in the case of slender body wakes (where the inviscid flow field does not vary much), but without much error in all cases where chemical reactions do not make the flow properties vary too rapidly with distance, we may rewrite Eq. (16) as

$$\frac{d}{dz}\left(\rho_m V_m A_m\right) \simeq \rho_i(z - \xi)V_i(z - \xi) \frac{dA\,(z - \xi)}{d(z - \xi)} \int_{z_0}^{z} \frac{\partial \phi}{\partial z}\, dz'$$

$$= \rho_i(z - \xi)V_i(z - \xi) \frac{dA\,(z - \xi)}{dz} , \qquad (17)$$

where ξ is now the value about which $(\partial\phi/\partial z)$ peaks. If the peak of $(\partial\phi/\partial z)$ is assumed infinitely sharp,

$$\frac{\partial\phi}{\partial(z - \xi)} = \delta(z - \xi) ,$$

and

$$\frac{\partial\phi}{\partial z} = \frac{d(z - \xi)}{dz}\,\delta(z - \xi) ,$$

confirming Eq. (7). Thus the wake mixing formulation using a sharp lag can be used to describe gradual mixing over a range small compared to the range over which the inviscid flow properties change significantly.

5. NUMERICAL RESULTS AND DISCUSSION

In the present section, a numerical solution of the wake mixing equations using a simplified set of an effective mixing boundary is presented and discussed. It should be stressed at the outset that the model is quite crude, and that in particular it does not account very well for what might be termed dynamic effects, such as drag between the mixed and unmixed components, the effects on pressure of mixing of the two components, and so forth. It is primarily what its name implies, i.e., a mixing model, and is intended for application to reacting wakes. However, the low speed non-reacting case is of interest, particularly in comparing its predictions with experimental results. The calculations were made to correspond to the case of a one centimeter diameter body traveling at some 10,000 feet/sec.

The mean wake boundary used in the numerical computations was obtained from a $\frac{1}{3}$ power law fitted to experiment. The lag function was approximated by a linear dependence of ξ on downstream distance over several wake segments, with the coefficients chosen to roughly match Eq. (15). The value of n in Eq. (15) over the various segments was estimated by reference to measured data on velocity decay in wakes[15], and $(\bar{\rho}/\rho_\infty)$ was estimated from a theoretical calculation by L. Hromas[16]. From such estimates, the lag was assumed to rise from 0.05z for z less than 50 body diameters to 0.1z between 50 and 150 diameters, and increasing to 0.5z for z greater than 300 diameters, with additive constants used to make ξ continuous. Beyond 500 diameters, ξ was approximated to reach a constant value corresponding to a lag time of approximately

two milliseconds, which is approximately the diffusion time for molecular diffusion from a one centimeter diameter sphere at a temperature of approximately 500°K and six percent atmospheric pressure.

The specification of the inviscid flow outside the turbulent core was obtained from a hemisphere cylinder solution also supplied by L. Hromas[16]. The free stream velocity in the numerical case used was about 9600 ft/s and the free stream density used was $(\rho_\infty/\rho_o) \sim 6 \times 10^{-2}$ corresponding to an altitude of 60,000 ft. An equilibrium gas was assumed in the numerical computations.

The results obtained by solving the one-dimensional channel flow with an effective mixing boundary are shown in Figs. (2) to (4).

Figure (2) shows the variation of the density ρ_m along the wake as well as the mean density $\bar{\rho}_u$ of the unmixed component. It can be seen that $\bar{\rho}_u$ becomes substantially greater than ρ_m as we proceed downstream. This leads to a relatively large ratio $\tau_\rho = \rho_u/\rho_m$ and the possibility of large relative fluctuations, as discussed in subsection 2-4. The ratio τ_ρ is plotted in Fig. (3) along with the unmixed volume fraction ϵ.

Finally, the relative root-mean square gas density fluctuation $(\rho'/\bar{\rho})$ is plotted in Fig. (4). The gas density fluctuation is obtained from the equation

$$\frac{\rho'}{\bar{\rho}} = \frac{1}{\bar{\rho}} \left\{ \epsilon \left(\rho_u^2 - \bar{\rho}_u^2 \right) + \epsilon(1 - \epsilon) \left(\bar{\rho}_u - \rho_m \right)^2 \right\} \qquad (18)$$

which accounts for fluctuations arising from inhomogeneity of the unmixed gas.

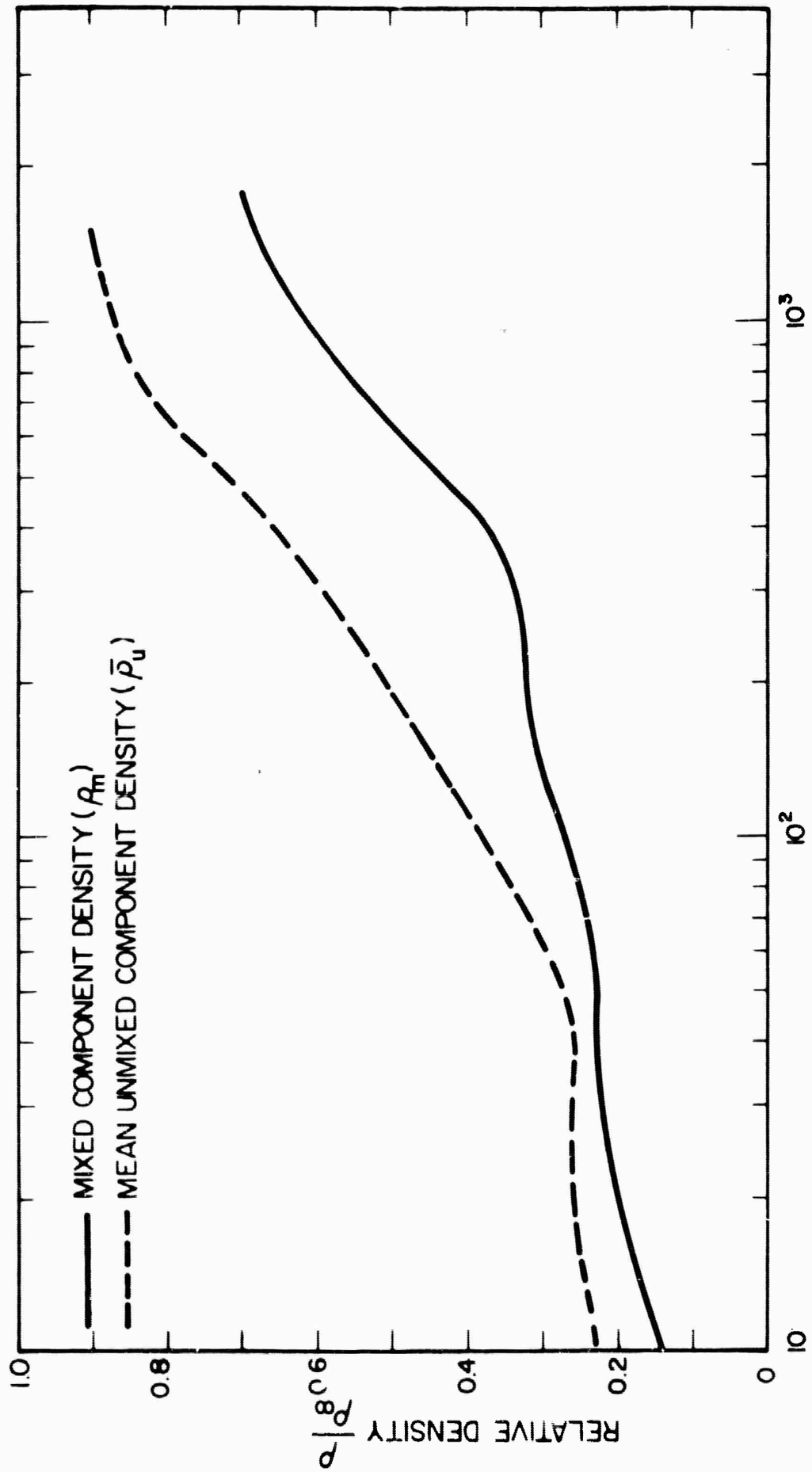

Fig. 2 Variation of mixed fluid density ρ_m and mean unmixed fluid density $\bar{\rho}_u$ with downstream distance in mixing with lag model.

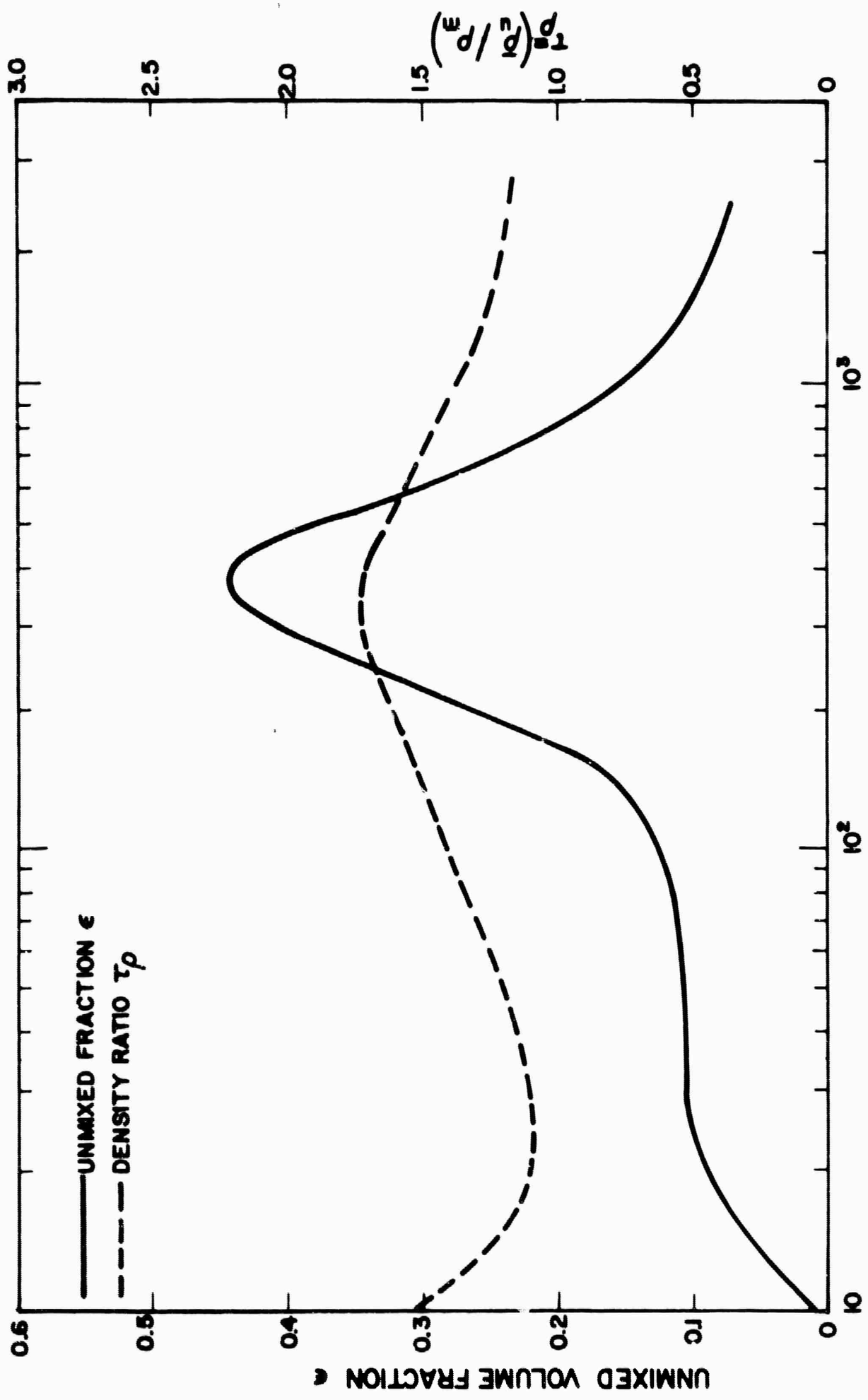

Fig. 3 Variation of ratio of mixed fluid density to mean unmixed fluid density and of unmixed volume fraction ε with downstream distance.

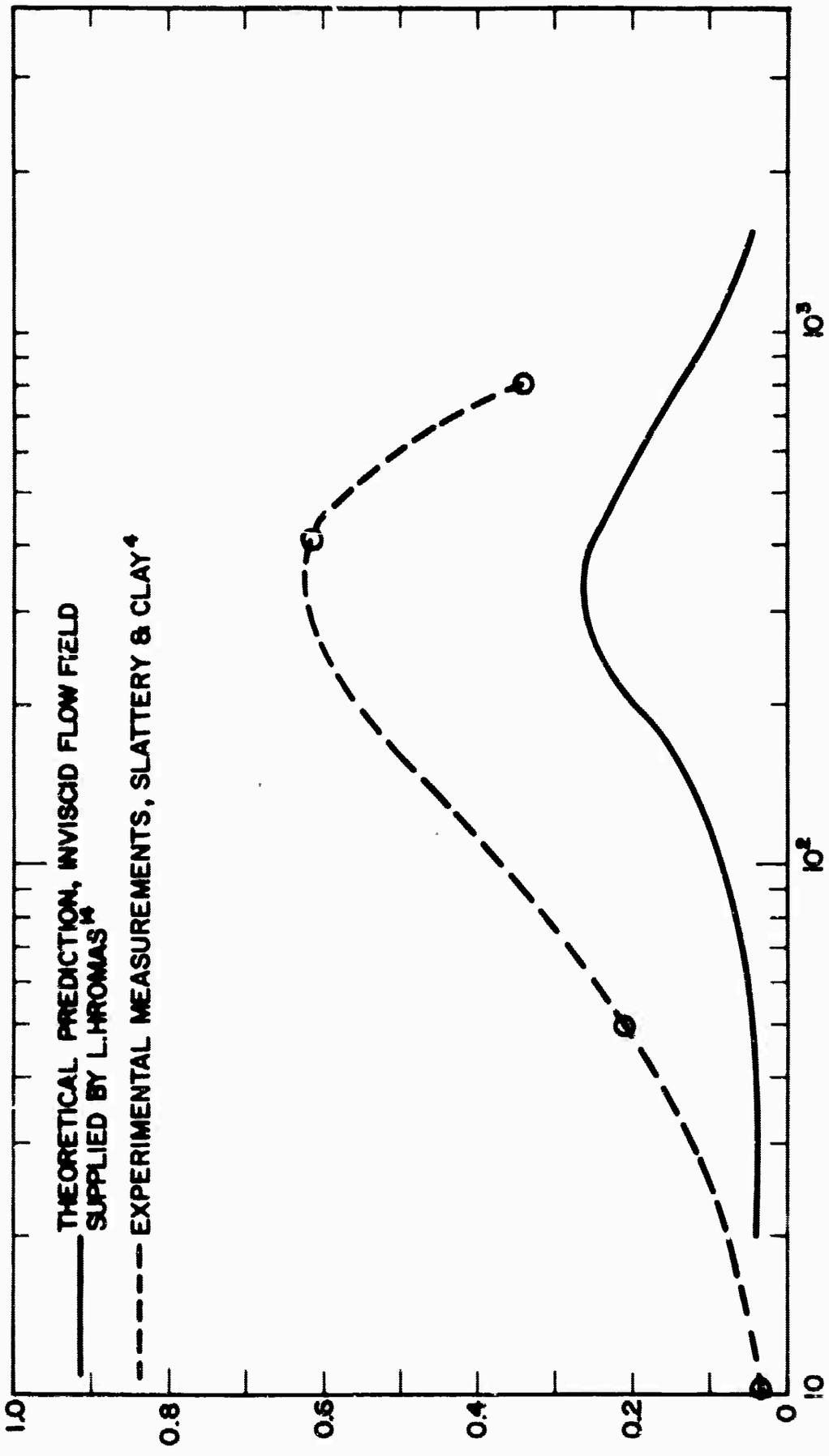

Fig. 4 Normalized mass density fluctuations along wake according to mixing with lag model. The experimental values shown have been recently revised sharply downward by the experimenters[12], although precise new values have not yet been made available.

The experimental results obtained at Lincoln Laboratory using schlieren methods agree with the predicted trend in the data, with a maximum observed amplitude around 0.1, for pellets traveling at approximately 8500 ft/sec. There appears to be some uncertainty, however, in the determination of the absolute magnitude of the density fluctuations which they measure.

The present paper has been based upon a characterization of wake mixing in terms of propagation of turbulence by small scale eddies, random convection of fluid elements within the turbulent core by large scale eddies, and dissipation of inhomogeneities by the combined action of distortion of fluid elements by turbulent motions and molecular diffusion. Such a characterization is consistent with the theory of (low-speed) turbulence as it exists today. It leads to a two-component wake model, in which one component consists of fluid which is relatively homogeneous and well mixed, while the other consists of fluid newly entrained into the turbulent core, and therefore retaining its unmixed properties. In the above model, the time delay in thorough (molecular) mixing of the latter component was represented by a lag time or lag distance.

A set of one-dimensional equations derived for such a model was developed, and the solution to a simplified version of those equations for a relatively low speed wake provided predictions of density fluctuations in the wake in good agreement with available experimental data.

The model developed above is intended for use in predicting the properties of turbulent wakes of high speed re-entry vehicles. For such wakes, the chemical reactions in the wake are of primary

importance, and the model requires extension and further refinement in order to provide reliable predictions for reacting wakes of high speed bodies, though it can be used for rough calculations by including reactions in the conservation equations for each specie of importance. In any event, the ability of the model to predict some of the features of wakes of low-speed bodies is an encouraging indication of the validity of the concept of a two-component wake.

Finally, the validity of the model could be more critically tested if measurements of both gas and electron density fluctuations in the wake were available, as the model makes definite predictions concerning the relation between those fluctuations.

APPENDIX A

In this appendix, a rough estimate of the magnitude of density oscillations which can be generated in a turbulent wake by dissipation and compressibility effects is provided.

The order of magnitude of the density fluctuations that could be generated in a turbulent wake may be estimated as follows: assume that the source of the fluctuations is dissipation or pV work and is generated by the fluctuating velocities rather than convection of cold high density gas into hot low density gas. Since there is no way to support large pressure fluctuations in a free flow like a wake, beyond a few downstream diameters (particularly for free stream velocities of 10^4 ft/sec or so), we assume that the fluctuations occur at constant pressure. For the case of fluctuations generated by viscous dissipation, it may be assumed that at most all the random kinetic energy per unit mass in a given unit mass of fluid will be converted into heat in generating a density change.*

The kinetic energy E_{turb} per unit mass in the random velocity field is approximately

$$E_{turb} = \frac{3}{2} u'^2 = \frac{3}{2} \left(\frac{u'^2}{\Delta V^2} \right) \left(\frac{\Delta V}{V_\infty} \right)^2 V_\infty^2 \qquad (A-1)$$

where u' is the root mean square turbulent velocity, ΔV is the wake velocity defect, and V_∞ is the free stream velocity.

*This is an upper limit by any standard.

The ratio $(u'^2/\Delta V^2)$ can be estimated to be roughly one-tenth beyond a few tens of body diameter. The approximate relation (see Section 2.1)

$$\left(\frac{\Delta V}{V_\infty}\right) \sim \left(\frac{1}{8} y_f^2\right)\left(\frac{\rho_\infty}{\rho}\right) \tag{A-2}$$

may be used for the wake velocity decay.

On the other hand, assuming that all the kinetic energy of turbulence is converted to heat, so that $c_p \Delta T \sim E_{turb}$, the resulting density change is

$$\frac{\Delta\rho}{\rho} - -\frac{\Delta T}{T} = -\frac{E_{turb}}{c_p T} = -\frac{3}{2}\left(\frac{u'}{\Delta V}\right)^2\left(\frac{\Delta V}{V_\infty}\right)^2\left(\frac{V_\infty^2}{h}\right) \tag{A-3}$$

where h is the local static enthalpy.

The ratio (V_∞^2/h) can be estimated by writing it in the form

$$\frac{V_\infty^2}{h} = \frac{V_\infty^2}{h_\infty + \frac{1}{2}\left(V_\infty^2 - V_w^2\right)} \approx \frac{2V_\infty}{\Delta V} \tag{A-4}$$

Finally, substituting Eqs. (A-1), (A-2) and (A-4) into (A-3), and assuming $y_f \sim (1/2)z^{1/3}$, the density fluctuation can be estimated as

$$\left|\frac{\Delta\rho}{\rho}\right| \sim 0.15 z^{-2/3}\left(\frac{\rho_\infty}{\rho}\right), \tag{A-5}$$

where $\bar{\rho}$ is the mean wake density. From the above equation, it is clear that at downstream distances of the order of one to several hundred body diameters, the density fluctuations due to dissipation can be expected to be of the order of 1% or less. The same estimates can be shown to result if it is assumed that the density fluctuations are generated by conversion of turbulence energy into pV work.

Thus, it is safe to assume that if fluctuations in density of the order of several tens percent do exist in the wake, as some evidence indicates, they must arise from a random mixing effect.

APPENDIX B

In the present appendix characteristic time scales for turbulent cascade or dissipation processes and for molecular diffusion are estimated and the validity of the mixing model used is discussed in terms of those time scales.

The present mixing model assumes that the rate of molecular mixing of fluid ingested into the turbulent core with the gases present therein is limited by the rate at which the turbulent velocity field breaks up fluid elements from a relatively large scale D to a scale at which molecular diffusion is highly effective. This implies that the time scale for this turbulent shredding is short compared to the time for molecular diffusion to erase inhomogeneities of scale D. Thus, a minimum requirement for the validity of the present model is:

$$\tau \ll \tau_{diff}(D) \tag{B.1}$$

where τ is the turbulence cascade time and τ_{diff} is the molecular diffusion time scale. The latter may be estimated as

$$\tau_{diff}(l) \sim \frac{l^2}{K} , \tag{B.2}$$

where K is a diffusion coefficient. If we choose the coefficient of ambipolar diffusion in air for K, we have:

$$K \sim 0.1\left(\frac{\rho_o}{\rho}\right)\left(\frac{T}{273}\right)^{1/2} . \qquad \text{(B. 3)}$$

The scale D may be assumed to be roughly a body diameter for the purposes of the present estimate. Substituting the above estimates in (B. 1), we can express it approximately as

$$\tau \ll 100D^2\left(\frac{\rho}{\rho_o}\right)\left(\frac{T}{273}\right)^{-1/2} . \qquad \text{(B. 4)}$$

The time τ has been assumed in the present paper to be roughly the same as the turbulent cascade time. The latter is approximately L/u', where L is the scale of the energy containing eddies of turbulence and u' is the root mean square turbulent velocity. Assuming approximately L to be equal to D, and u' to be one half the mean wake velocity ΔV, (B. 4) becomes:

$$2\left(\frac{V_\infty}{\Delta V}\right)\left(\frac{D}{V_\infty}\right) \ll 100D^2\left(\frac{\rho}{\rho_o}\right)\left(\frac{T}{273}\right)^{-1/2} . \qquad \text{(B. 5)}$$

Assuming now that the density ρ can be estimated in terms of the free stream density as

$$\left(\frac{\rho}{\rho_\infty}\right) \simeq \left(\frac{T}{T_\infty}\right)^{-1} \simeq \left(\frac{T}{273}\right)^{-1} ,$$

and using the same approximate velocity decay law as was used in Section 2 of the text, namely

$$\left(\frac{\Delta V}{V_\infty}\right) \sim \frac{1}{8} y_f^{-2} \left(\frac{\rho_\infty}{\rho}\right)$$

we can express the inequality (B. 5) in terms of a free stream Reynolds number:

$$R_{e_\infty} = \frac{V_\infty \rho_\infty D}{\mu_\infty} , \quad \mu_\infty = \text{free stream viscosity.}$$

After simple substitutions, the result is:

$$R_{e_\infty} \gg 4Z^{2/3} \left(\frac{T}{273}\right)^{1/2} , \qquad \text{(B. 6)}$$

where T has been taken throughout as some mean wake temperature.

The above inequality is easily satisfied at least to downstream distances of a thousand body diameters for Reynolds numbers for which the wake becomes turbulent close to the body, i. e., for Reynolds numbers of fifty thousand or more.

The requirement for the validity of the mixing model in terms of the relative scales for turbulent and laminar dissipation is somewhat more stringent than indicated by Eq. (B. 6). Indeed, the turbulent dissipation time scale should be small compared to the molecular dissipation time scale over the major portion of the dissipation "history", so to speak, of a fluid element. In other words an

inequality of the form (1) must hold for times τ and τ_{diff} based on scales smaller than D, that is, as the cascade process is occurring. It can be shown that this requirement leads to a required Reynolds number about one order of magnitude greater than indicated by (B. 6).

REFERENCES

1. Lees, L. and Hromas, L., "Turbulent Diffusion in the Wake of a Blunt Nosed Body at Hypersonic Speeds," J. Aerospace Sci. 29, 976 (1962).

2. Webb, W., and Hromas, L., "Turbulent Diffusion of a Reacting Gas in the Wake of a Sharp Nosed Body at Hypersonic Speeds," Space Technology Laboratories, Inc., Report No. 6130-6362-RU-000 (April 1963).

3. Webb, W., and Hromas, L., "Turbulent Diffusion of a Reacting Wake," AIAA Preprint 64-42 (January 1964).

4. Proudian, A. and Feldman, S., "Some Theoretical Predictions of Mass and Electron Density Oscillations Based on a Simple Model for Turbulent Wake Mixing," AIAA Preprint 64-21 (January 1964).

5. Proudian, A., "A New Quasi-One Dimensional Model for Turbulent Wake Mixing," Heliodyne Corp. Research Note 13 (January 1964).

6. Clay, W.G., Hermann, J., and Slattery, R.E., "The Measured Statistical Properties of Wake Turbulence Behind Hypervelocity Spheres," AMRAC Proceedings X, 27 (April 1964).

7. Clay, W.G., Lincoln Laboratory, MIT, Private Communication (May 1964).

8. Townsend, A.A., The Structure of Turbulent-Shear Flow, Cambridge University Press, Cambridge (1956).

9. Hinze, J.O., Turbulence, McGraw-Hill, New York (1959).

10. Corrsin, S., and Kistler, A.A., "Free-Stream Boundaries of Turbulent Flows," NACA Report 1244, Washington (1955).

11. Batchelor, G.K., "The Effect of Homogeneous Turbulence on Material Lines and Surfaces," Proc. Roy Soc. London, 213, 349 (1952).

12. Clay, W.G., Lincoln Laboratory, MIT, Private Communication (November 1964).

13. Lin, S.C., and Hayes, J.E., "A Quasi One-Dimensional Model for Chemically Reacting Turbulent Wakes of Hypersonic Objects," Research Report 157, Avco-Everett Research Laboratory (July 1963).

14. Onsager, L., "Statistical Hydrodynamics," Nuovo Cimento Suppl. 2, 279 (1949).

15. Slattery, R.E., and Clay, W.G., "Measurement of Turbulent Transition, Motion, Statistics and Gross Radial Growth Behind Hypervelocity Objects," Phys. Fluids, 5, 849 (1962).

16. Hromas, L., Space Technology Laboratories, Inc., Private Communication (October 1963 and June 1964).